Linear Form of the Radiative Transfer Equation Revisited

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Outline

- Smith's linear form (*Applied Optics, 1991*) vs. its dual form.
- The linear form with *exact* analytical Jacobians vs. the tangent-linear and adjoint models.
- The linear form with exact analytical Jacobians vs. the linear form with *inexact* analytical Jacobians.
- Why use the forward model with exact analytical Jacobians for physical retrieval and data assimilation?
- Summary
- Future work

Bill Smith’s Achievement in Physical Retrieval

- Bill has been pioneering the physical retrieval of temperature and absorbing constituent profiles from the radiance spectra since the late ’60s.
- His most recent linear form of the RTE was published in the landmark paper in *Applied Optics* (1991).
- This *monochromatically approximate* linear form and its variant have been still used in the physical retrieval [e.g. Ma *et al*., 2000, Li *et al*., 2001].
- His work inspired Huang *et al.* (*Applied Optics*, 2002) to successfully derive the linear form with <u>*exact*</u> analytical Jacobians for the widely-used McMillin-Fleming-Eyre-Woolf type of forward models (e.g. RTTOV, RTIASI).
- In this talk I will prove that there exists the *dual* representation of Smith’s linear form (1991), and show some mathematically interesting outcomes derived from this dual form.

Smith's Linear Form (1991) vs. its Dual Form

$$R_{\nu}^{obs} = B_{\nu}(p_s)\tau_{\nu}(p_s) - \int_0^{P_S} B_{\nu}(p)\, d\tau_{\nu}(p)$$

$$R_{\nu}^{0} = B_{\nu}^{0}(p_s)\tau_{\nu}^{0}(p_s) - \int_0^{p_s} B_{\nu}^{0}(p)\, d\tau_{\nu}^{0}(p)$$

$$\delta R_{\nu} \equiv R_{\nu}^{obs} - R_{\nu}^{0}$$

$$\delta B_{\nu}(p) \equiv B_{\nu}(p) - B_{\nu}^{0}(p)$$

$$\delta \tau_{\nu}(p) \equiv \tau_{\nu}(p) - \tau_{\nu}^{0}(p)$$

Smith's Linear Form

$$\delta R_{\nu} = B_{\nu}(p_s)\delta\tau_{\nu}(p_s) + \delta B_{\nu}(p_s)\tau_{\nu}^{0}(p_s) - \int_0^{p_s} B_{\nu}(p)\, d\left[\delta\tau_{\nu}(p)\right] - \int_0^{p_s} \delta B_{\nu}(p)\, d\tau_{\nu}^{0}(p)$$

Its Dual Form

$$\delta R_{\nu} = \delta B_{\nu}(p_s)\tau_{\nu}(p_s) + B_{\nu}^{0}(p_s)\delta\tau_{\nu}(p_s) - \int_0^{p_s} \delta B_{\nu}(p)\, d\tau_{\nu}(p) - \int_0^{p_s} B_{\nu}^{0}(p)\, d\left[\delta\tau_{\nu}(p)\right]$$

$$\delta R_v = B_v(p_s)\,\delta\tau_v(p_s) + \delta B_v(p_s)\,\tau_v^0(p_s) - \int_0^{p_s} B_v(p)\,d\left[\delta\tau_v(p)\right] - \int_0^{p_s} \delta B_v(p)\,d\tau_v^0(p)$$

$$\delta R_v = \delta B_v(p_s)\,\tau_v(p_s) + B_v^0(p_s)\,\delta\tau_v(p_s) - \int_0^{p_s} \delta B_v(p)\,d\tau_v(p) - \int_0^{p_s} B_v^0(p)\,d\left[\delta\tau_v(p)\right]$$

$$\delta B_v(p) = \frac{\partial B_v(T^0(p))}{\partial T(p)}\,\delta T(p) \equiv \beta_v^0(p)\,\delta T(p)$$

$$\delta\tau_v(p) \approx \tau_v^0(p) \sum_{i=1}^{N} \delta U_i(p)\,\frac{d\ln\tau_{v_i}^0(p)}{dU_i^0(p)}$$

$$U_i(p) \equiv \frac{1}{g}\int_0^p q_i(p')\,dp'$$

$$d\tau_v^0(p) = \tau_v^0(p) \sum_{i=1}^{N} d\ln\tau_{v_i}^0(p)$$

$$\frac{d\ln\tau_{v_i}^0(p)}{dU_i^0(p)}\,dT(p) = d\ln\tau_{v_i}^0(p)\,\frac{dT(p)}{dU_i^0(p)},$$

$$\tau_v(p_s) \approx \tau_v^0(p_s)$$

$$dU_i(p)\,\frac{d\ln\tau_{v_i}^0(p)}{dU_i^0(p)} = \frac{dU_i(p)}{dU_i^0(p)}\,d\ln\tau_{v_i}^0(p)$$

$$\delta R_v \approx \beta_v^0(p_s)\,\tau_v^0(p_s)\,\delta T_s - \sum_{i=1}^{N}\int_0^{p_s} \beta_v^0(p)\,\tau_v^0(p)\,\delta T(p)\,d\ln\tau_{v_i}^0(p) + \sum_{i=1}^{N}\int_0^{p_s} \beta_v^0(p)\,\tau_v^0(p)\,\delta U_i(p)\,\frac{dT(p)}{dU_i^0(p)}\,d\ln\tau_{v_i}^0(p)$$

$$\delta R_v \approx \beta_v^0(p_s)\,\tau_v^0(p_s)\,\delta T_s - \sum_{i=1}^{N}\int_0^{p_s} \beta_v^0(p)\,\tau_v^0(p)\,\delta T(p)\,\frac{dU_i(p)}{dU_i^o(p)}\,d\ln\tau_{v_i}^0(p) + \sum_{i=1}^{N}\int_0^{p_s} \beta_v^0(p)\,\tau_v^0(p)\,\frac{dT^0(p)}{dU_i^0(p)}\,\delta U_i(p)\,d\ln\tau_{v_i}^0(p)$$

$$\delta R_\nu \approx \beta_\nu^0(p_s)\tau_\nu^0(p_s)\delta T_s - \sum_{i=1}^{N}\int_0^{p_s}\beta_\nu^0(p)\tau_\nu^0(p)\delta T(p)\,d\ln\tau_{\nu_i}^0(p) + \sum_{i=1}^{N}\int_0^{p_s}\beta_\nu^0(p)\tau_\nu^0(p)\delta U_i(p)\frac{dT(p)}{dU_i^0(p)}\,d\ln\tau_{\nu_i}^0(p)$$

$$\delta R_\nu \approx \beta_\nu^0(p_s)\tau_\nu^0(p_s)\delta T_s - \sum_{i=1}^{N}\int_0^{p_s}\beta_\nu^0(p)\tau_\nu^0(p)\delta T(p)\frac{dU_i(p)}{dU_i^o(p)}\,d\ln\tau_{\nu_i}^0(p) + \sum_{i=1}^{N}\int_0^{p_s}\beta_\nu^0(p)\tau_\nu^0(p)\frac{dT^0(p)}{dU_i^0(p)}\delta U_i(p)\,d\ln\tau_{\nu_i}^0(p)$$

The effective temperature profile of the *i* *th* absorbing gas:

$$\delta T_i(p) \equiv \delta T(p) - \delta U_i(p)\frac{dT(p)}{dU_i^0(p)}$$

The effective temperature profile of the *i* *th* absorbing gas:

$$\delta T_i(p) \equiv \delta T(p)\frac{dU_i(p)}{dU_i^0(p)} - \delta U_i(p)\frac{dT^0(p)}{dU_i^0(p)}$$

Final Linear Form

$$\delta R_\nu \approx \beta_\nu^0(p_s)\tau_\nu^0(p_s)\delta T_s - \sum_{i=1}^{N}\int_0^{p_s}\beta_\nu^0(p)\delta T_i(p)\tau_\nu^0(p)\,d\ln\tau_{\nu_i}^0(p)$$

$$\delta T_i(p) \equiv \delta T(p) - \delta U_i(p)\frac{dT(p)}{dU_i^0(p)}$$

$$U_i(p) = U_i^0(p) + \frac{dU_i^0(p)}{dT(p)}\left[T(p) - T_i(p)\right]$$

Same!

$$\delta T_i(p) \equiv \delta T(p)\frac{dU_i(p)}{dU_i^0(p)} - \delta U_i(p)\frac{dT^0(p)}{dU_i^0(p)}$$

$$\left[T^0(p) - T(p)\right]\frac{dU_i(p)}{dT^0(p)} + U_i(p)$$

$$= U_i^0(p) + \frac{dU_i^0(p)}{dT^0(p)}\left[T^0(p) - T_i(p)\right]$$

A special case: $T(p) = T^0(p)$

$$\Rightarrow$$

$$U_i(p) = U_i^0(p) + \frac{dU_i^0(p)}{dT(p)}\left[T(p) - T_i(p)\right]$$

The general case: $T(p) \neq T^0(p)$

$$\Rightarrow$$

$$U_i(p) = \frac{1}{T^0(p) - T(p)} \times \int_0^p \left\{ U_i^0(p) + \frac{dU_i^0(p')}{dT^0(p')}\left[T^0(p') - T_i(p')\right] \right\} \frac{dT^0(p')}{dp'} \, dp'$$

The retrieval quality of absorbing gas profiles depends on the quality of temperature first guess!

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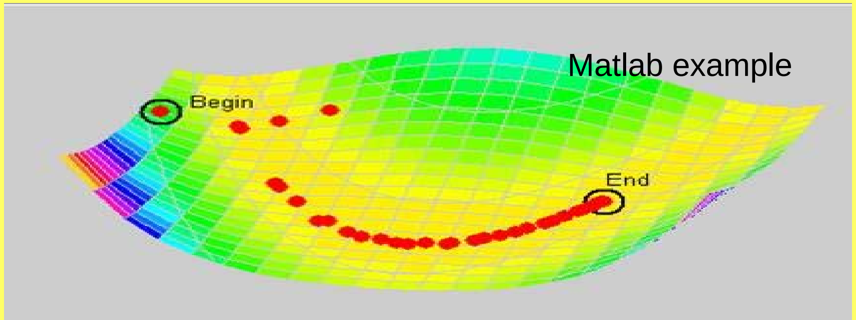
Comparison of linear forms of the radiative transfer equation with analytic Jacobians

Bormin Huang, William L. Smith, Hung-Lung Huang, and Harold M. Woolf

$$R_\nu = \varepsilon_{\nu s} B_\nu(T_s)\tau_\nu(p_s) - \int_0^{p_s} B_\nu[T(p)] \frac{d\tau_\nu(p)}{dp} dp + r_{\nu s}\tau_\nu(p_s)\int_0^{p_s} B_\nu[T(p)] \frac{d\tau_\nu{}^*(p)}{dp} dp + R_\nu{}^{\text{sun}}\tau_\nu{}^{1+\sec\Theta}(p_s) r_{\nu s}{}^{\text{sun}},$$

$$\tau_\nu(p_j) = \exp\left\{\sum_{k=1}^{j}\left[\sum_{l_f=1}^{m_f} a_{\nu l_f k}{}^{\text{fixed}} X_{l_f k}{}^{\text{fixed}} + \sum_{l_w=1}^{m_w} b_{\nu l_w k}{}^{\text{water}} X_{l_w k}{}^{\text{water}} + \sum_{l_o=1}^{m_o} b_{\nu l_o k}{}^{\text{ozone}} X_{l_o k}{}^{\text{ozone}}\right]\right\},$$

$$\delta R_\nu = W_{T_s}{}^0 \delta T_s + \sum_{j=1}^{L} W_T{}^0(p_j)\delta T(p_j) + \sum_{i=1}^{N}\sum_{j=1}^{L} W_{q_i}{}^0(p_j)\delta q_i(p_j) + W_{\varepsilon_{\nu s}}{}^0 \delta\varepsilon_{\nu s} + W_{r_{\nu s}}{}^0 \delta r_{\nu s} + W^0_{r_{\nu s}^{\text{sun}}} \delta r_{\nu s}{}^{\text{sun}},$$



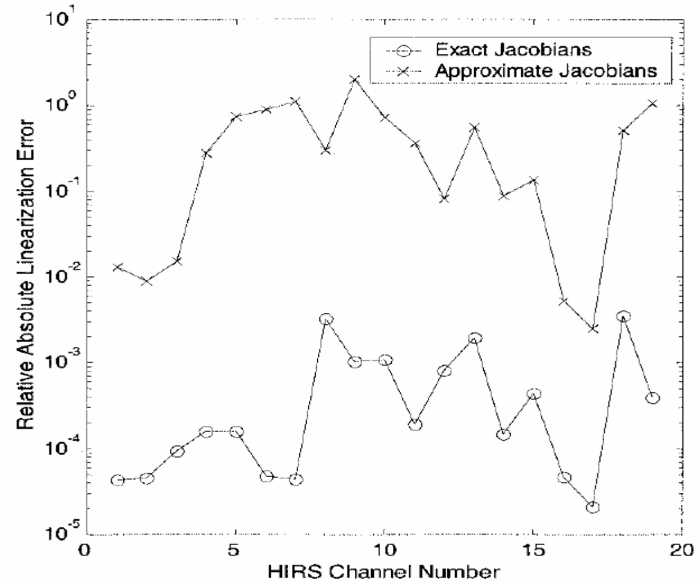


Fig. 1. Comparison of the relative absolute linearization errors between the linear form with exact analytic Jacobians and the linear form with approximate analytic Jacobians.

Table 1. Comparison between the Two Linear Forms in Eqs. (44) and (80)[a]

HIRS Channel Number	LHS of Eqs. (44) and (80)	RHS of Eq. (80)	RHS of Eq. (44)
1	0.23273	0.23274	0.22972
2	0.21944	0.21945	0.21749
3	0.21535	0.21537	0.21206
4	0.18574	0.18577	0.13306
5	0.18711	0.18708	0.04729
6	0.20811	0.20810	0.01729
7	0.22648	0.22649	−0.02717
8	0.23675	0.23753	0.30907
9	0.16291	0.16308	−0.16596
10	0.21283	0.21260	0.05626
11	0.20545	0.20541	0.13039
12	0.20825	0.20842	0.19091
13	0.15437	0.15407	0.06667
14	0.20446	0.20449	0.22248
15	0.20524	0.20515	0.17713
16	0.21464	0.21465	0.21352
17	0.23486	0.23486	0.23545
18	0.05087	0.05105	0.02483
19	0.07582	0.07585	−0.00474

[a] The RHS of Eq. (44) is the linear form with approximate analytic Jacobians, whereas the RHS of Eq. (80) is the linear form with exact analytic Jacobians. Equations (44) and (80) have the same LHS, which is calculated by the forward model and represents the true brightness temperature difference in the two conditions.

The exact analytical Jacobians approach vs. the tangent-linear & adjoint approach

1. **Same numerical precision!**

2. **The exact analytical Jacobian approach is several times faster !!**

Example:

$$V_p(T_d) = C\,(\frac{T_0}{T_d})^A\ e^{(A+B)\,(1-\frac{T_0}{T_d})}$$

Compute

$$\delta V_p = \frac{dV_p}{dT_d}\ \delta T_d$$

```
function J = Exact_Analytic_Jacobian(Td,A,B,C,To)
J = 1/C * (To/Td)^(-A) * exp(-(A+B) * (1-To/Td)) / (-A*Td + To*(A+B)) * Td^2;
```

```
; THE FORWARD MODEL
pro calculate_Vp_FWD, Td, $ ; Input Vp     ; Output
  common Constants
  Ratio = To / Td
  X = C * ( Ratio^A )
  Y = EXP( (A+B) * ( ONE - Ratio ) )
  Vp = X * Y
end

; THE TANGENT-LINEAR MODEL
pro calculate_Vp_TL, Td,    $ ; Input
                     Td_TL, $ ; Input
                     Vp_TL    ; Output
  common Constants
  Ratio = To / Td
  X = C * ( Ratio^A )
  Y = EXP( (A+B) * ( ONE - Ratio ) )
  Ratio_TL = ( -ONE * To / Td^2 ) * Td_TL
  X_TL = C * A * ( Ratio^(A-ONE) ) * Ratio_TL
  Y_TL = -(A+B) * Y * Ratio_TL
  Vp_TL = ( X_TL * Y ) + ( X * Y_TL )
end

; THE ADJOINT MODEL
pro calculate_Vp_AD, Td,    $ ; Input
                     Vp_AD, $ ; Input and Output
                     Td_AD    ; Input and output
  common Constants
  Ratio = To / Td
  X = C * ( Ratio^A )
  Y = EXP( (A+B) * ( ONE - Ratio ) )
  X_AD = Y * Vp_AD
  Y_AD = X * Vp_AD
  Vp_AD = ZERO
  Ratio_AD = ( -(A+B) * Y ) * Y_AD
  Y_AD = ZERO
  Ratio_AD = Ratio_AD + ( C * A * ( Ratio^(A-ONE) ) ) * X_AD
  X_AD = ZERO
  Td_AD = Td_AD + ( -ONE * To / Td^2 ) * Ratio_AD
  Ratio_AD = ZERO
end
```

Source: Paul van Delst

Why Use Exact Analytical Jacobians in Retrieval and Data Assimilation?

- Physical Retrieval (1D-Var):

$$Cost(X) = \left\| R^{obs} - R(X) \right\| \quad or \quad Cost(X) = \left\| R^{obs} - R(X) \right\| + \lambda \left\| X - X^{0} \right\|$$

- NWP Data Assimilation (3D/4D-Var):

$$Cost(X,...) = Atmospheric\ Dynamic\ Term(X,...) + \left\| R^{obs} - R(X) \right\|$$

~ ideal choice for multispersptral sounders (e.g. HIRS, GOES, MODIS)
~ an underdetermined problem (with respect to a typical forward model)

$$Cost(X,...) = Atmospheric\ Dynamic\ Term(X,...) + \left\| X^{retrieval} - X \right\|$$

~ ideal choice for hyperspctral sounders (e.g. AIRS, IASI, GIFTS)
~an overdetermined problem (with respect to a typical forward model)

- **Atmospheric dynamic term is basically governed by the Navier-Stokes equation, which has no analytical Jacobians. Thus, its tangent-linear/adjoint models are needed for data assimilation.**
- **The exact analytical Jacobians** for t**he widely-used McMillin-Fleming-Eyre-Woolf type of radiative transfer/forward models (e.g. RTTOV, RTIASI) are derivable (Huang et al., Applied Optics, 2002).**

Summary

1. **The classical derivation of the linear form of the RTE by Smith *et al*. (1991) is reviewed. Its dual form is derived.**
2. **The original linear form appears to be a special case of its dual form when the temperature first guess happens to be the true temperature profile.**
3. **Linear forms with *inexact* analytic Jacobians make retrieval results unreliable!**
4. **The *exact* analytic Jacobians implementation is an efficient alternative to the tangent-linear/adjoint models for hyperspectral retrieval and data assimilation problems with the widely-used McMillin-Fleming-Eyre-Woolf type of forward models (e.g. RTTOV, RTIASI).**

Future Work

The Remote Sensing GENOME (Geometrical Exploration of Nonlinear Optimization in Measurement Environment) Project:

Unveiling the Radiance Hyperspace for Quantifying Geophysical Retrievals

The remote sensing GENOME project aims to solve the following long-standing fundamental problems in passive remote sensing

Given a sensor specification, its forward model and exact Jacobians, to quantify

- the information content of each channel, *i.e.,* the *best expected* contribution from each channel to the retrieval of temperature and absorbing gases at each pressure level,
- the information content of a sensor, *i.e.,* the *best expected* retrieval accuracy that sets the statistical limit for *all* retrieval algorithms,
- the error of a fast model in terms of the degradation of the *best expected* retrieval accuracy, as compared to its LBL counterpart,
- the impact of sensor noise level on the *best expected* retrieval accuracy,
- the "first guess tolerance" – a safety measure beyond which no retrieval algorithm can reach the *best expected* retrieval, and
- the "retrieval efficiency" – a robustness measure for any retrieval algorithm, as compared to the *best expected* retrieval.

Applications:

- *Lossy compression retrieval impact studies:*

  to conclude the retrieval degradation (due to lossy compression) by the best expected retrieval accuracy that sets the limit for *all* possible retrieval algorithms.

- *Optimal channel selection for retrieval with partial channels:*

  to relieve the computational burden in retrieval and data assimilation with the minimum retrieval degradation for hyperspectral sounders (e.g. AIRS, IASI, GIFTS).

- *Future sensor design & trade-off study for risk reduction:*

  to assess the information content (the *best expected* retrieval accuracy) of a sensor designed with various spectral ranges, ILS resolutions, and noise levels.

Different living species have different genomes. So do different sensors!